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Employing the genetic algorithm to estimate the bivariate Weibull Distribution parameters and comparing it with the CME method*

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Abstract:

The Bivariate Weibull Distribution (BVD) is widely used in reliability analysis to model dependent failures. Due to its complexity, parameter estimation poses significant challenges. This study applies Maximum Likelihood Estimation (MLE), Correlation-Based Estimation (CME), a Genetic Algorithm (GA) approach, and the Pseudo Maximum Likelihood Estimator (PMLE), all implemented in R. PMLE is introduced as a practical alternative when the full joint likelihood is analytically intractable, offering estimation based on simplified marginal components. Performance was evaluated using both simulated data (500 iterations, sample sizes 100 and 150). Results based on mean squared error (MSE) show that GA consistently outperforms MLE, CME, and PMLE, particularly in estimating shape, scale, and dependence parameters. GA proves to be a robust alternative for complex models, and future work may focus on hybridizing GA with MLE for improved estimation accuracy. The real-world data from the air quality dataset, modeling wind-speed and temperature dependence, further supports the theoretical model. The combined behavior of temperature and wind speed is well captured by the bivariate Weibull distribution, which is further strengthened by an interaction term controlled by θ. The model's ability to explain dependent data from the actual world is validated by the fitted parameters, which offer significant insight into the nature of atmospheric variability

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توظيف الخوارزمية الجينية (GA) لتقدير معلمات توزيع ويبل ذي المتغيرين، ومقارنتها مع طريقة (CME)*

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المستخلص

يعد توزيع ويبل ذي المتغيرين (BVD) من التوزيعات الواسعة الاستخدام في تحليل ونمذجة بيانات الفشل. ولكن من الصعب إيجاد المقدرات لكون الدالة معقدة، لذا تم في هذا البحث تطبيق طريقة الإمكان الأعظم(MLE)، وطريقة (CME)وتوظيف الطريقة الخوارزمية الجينية لغرض تقدير المعلمات و دالة المعولية. ولتحقيق ذلك، تم استخدام برنامج R في كتابة برنامج المحاكاة لغرض المقارنة بين طرائق التقدير، وذلك بتكرار مساوي 500 مرة وحجوم عينات مختلفة(N=100,150) و كذلك تم استخدام بيانات حقيقية وهي بيانات جودة الهواء (air quality) لمدينة (New York)، وذلك لنمذجة المتغيرات التي تمثل كل من سرعة الرياح ودرجة الحرارة. و تم التوصل الى ان طريقة الخوارزمية الجينية هي الأفضل في تقدير معلمات الشكل والقياس ومعلمة الاعتماد مقارنة بكل من طريقة MLE و CME من خلال المقياس الإحصائي متوسط مربعات الخطأ (MSE) . ولقد أثبتت طريقة GA نتائجها بفاعلية كبيرة خصوصًا في الحالات المعقدة، وقد يركز العمل المستقبلي على دمج كل من طريقة الخوارزمية الجينية (GA) مع طريقة (MLE). ومن خلال البيانات الحقيقية المستخلصة من مجموعة بيانات جودة الهواء تبين انه هناك ارتباط سالب بين كل من سرعة الرياح ودرجة الحرارة وقد تمكنا من نمذجة هذه البيانات باستخدام توزيع ويبل ذي المتغيرين والذي يصف العلاقة بين سرعة الرياح ودرجة الحرارة والذي يعبر بدقة عن السلوك المشترك لهذين العاملين وذلك من خلال المعلمة θ وقد اظهرت القيم المقدرة للمعاملات مدى قدرة النموذج على تفسير البيانات المرتبطة من الواقع، مما يوفر فهماً لطبيعة التغيرات في الغلاف الجوي.

الكلمات المفتاحية: توزيع ويبل ذي المتغيرين، تقدير المعلمات، الخوارزمية الجينيةGA.

1. Introduction:

The appeal of the Weibull distribution in reliability analysis stems from its capacity to model several different failure rates. The univariate form is widely used; however, real-world systems typically have dependent components which makes the Bivariate Weibull Distribution (BVD) more appropriate for capturing joint lifetimes (Mahdi & Gupta, 2011). X ~ Weibull (α_1, β_1) and Y ~ Weibull (α_2, β_2) are the shape and scale parameters of the bivariate Weibull distribution, respectively The parameter θ is Dependence Parameter captures the degree and nature of dependence between the two variables. When θ =0, the joint distribution reduces to the product of two independent Weibull distributions; that is, X and Y are

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statistically independent. As θ increases from 0 to 1, the level of negative dependence between X and Y increases. the result, maximal positive dependency and independence are represented by θ = 0 and θ = 1, respectively.

2. Bivariate Weibull distribution:

Bivariate Weibull distribution can address the life of a system exhibiting 2-dimensional characteristics in risk and reliability engineering. Given that the bivariate Weibull distribution contains more parameters than the univariate Weibull distribution, its usefulness has been hampered by its difficulties with parameter estimation. Taking into account a certain bivariate Weibull distribution model structure, (Yuan,2018)

A general bivariate Weibull model suggested by (Lu and Bhattacharyya (1990)) has the following form for its joint reliability function:

$$\bar{F}(x,y) = \exp\left[-\left\{\left(\frac{x}{\beta_1}\right)^{\alpha_1} + \left(\frac{y}{\beta_2}\right)^{\alpha_2} + \delta w(x,y)\right\}\right] \quad ...(1)$$

where $\alpha_i > 0$, $\beta_i > 0$, $\delta \geq 0$, for i=1,2 Different forms for the function w(x,y) yield different families of bivariate Weibull models. The specific form focused on in this paper is:

$$w(x,y) = \left(\frac{x}{\beta_1}\right)^{\alpha_1}\left(\frac{y}{\beta_2}\right)^{\alpha_2} \quad ...(2)$$

the bivariate Weibull reliability function becomes:

$$\bar{F}(x,y) = P(X > x, Y > y) = \exp\left[-\left\{\left(\frac{x}{\beta_1}\right)^{\alpha_1} + \left(\frac{y}{\beta_2}\right)^{\alpha_2} + \delta\left(\frac{x}{\beta_1}\right)^{\alpha_1}\left(\frac{y}{\beta_2}\right)^{\alpha_2}\right\}\right] \quad ...(3)$$

where x > 0, y > 0, $\alpha_1 > 0$, $\alpha 2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, and $0 \leq \delta \leq 1$.

Let us define the transformed variables:

$$U = \left(\frac{X}{\beta_1}\right)^{\alpha_1}, \quad V = \left(\frac{Y}{\beta_2}\right)^{\alpha_2} \quad ...(4)$$

Then, the joint reliability function of (U, V) becomes that of the bivariate exponential distribution introduced by (Gumbel (1960)). He showed If $\theta = 0$, then the correlation ρ=0, If θ=1, then ρ=−0.40365 In general, ρ is a decreasing function of δ, so it is non-positive.

for Bivariate Weibull Distribution we differentiating the reliability function in equation (3) to obtained the joint probability density function (Hyoung-Moon Kim et al,2024):

$$f(x,y)=\frac{\alpha_1}{\beta_1}\left(\frac{x}{\beta_1}\right)^{\alpha_1-1}\cdot\frac{\alpha_2}{\beta_2}\left(\frac{y}{\beta_2}\right)^{\alpha_2-1}\cdot\exp\left[-\left(\left(\frac{x}{\beta_1}\right)^{\alpha_1}+\left(\frac{y}{\beta_2}\right)^{\alpha_2}+\theta\left(\frac{x}{\beta_1}\right)^{\alpha_1}\left(\frac{y}{\beta_2}\right)^{\alpha_2}\right)\right]$$

$$\times\left[\left(1+\theta\left(\frac{x}{\beta_1}\right)^{\alpha_1}\right)\left(1+\theta\left(\frac{y}{\beta_2}\right)^{\alpha_2}\right)-\theta\right] \quad ...(5)$$

where:
$\alpha_{1,}$ $\alpha_2 > 0$ are shape parameters.
$\beta_1, \beta_2 > 0$ are scale parameters.
$(0 \le \theta \le 1)$ is the dependence (association) parameter between (X) and (Y).

3. Parameters explanation:

3-1. Shape Parameters (α1,α2);

The shape parameters determine the form of the marginal distributions:

- α_1 controls the tail behavior and skewness of the variable X (e.g., temperature).
- α_2 governs the same properties for variable Y (e.g., wind speed).

Higher values of α lead to less skewed distributions, while values closer to 1 result in more heavy-tailed or skewed behavior. When α=1, the Weibull reduces to an exponential distribution.

3-2. Scale Parameters (β1, β2)

The scale parameters influence the spread or location of the marginal:

- β_1 stretches or compresses the distribution of X.
- β_2 does the same for Y.

They are analogous to a unit of measurement or central tendency (though not identical to the mean), and larger values of β generally correspond to higher expected values.

3-3. Dependence Parameter (*θ*)

The parameter **θ** is Dependence Parameter captures the degree and nature of dependence between the two variables:

- When **θ** =0, the joint distribution reduces to the product of two independent Weibull distributions; that is, X and Y are statistically independent.
- As **θ** increases from 0 to 1, the level of negative dependence between X and Y increases.
- The correlation coefficient ρ becomes more negative as **θ** increases, reaching approximately −0.40365 when **θ** =1.

Because BVD does not have a closed-form for estimating the parameters because its structure is complex, classical methods such as Maximum

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Likelihood Estimation (MLE) or Correlation-Based Estimation (CME) suffer from a lack of intensive computation despite yielding consistent results (Kundu & Dey, 2009).

These issues have prompted researchers to turn their attention to heuristic approaches like the Genetic Algorithm (GA) that are known to perform well on complicated optimization tasks (AL-Wahid, 2017).

This research evaluates the efficiency of MLE, CME and GA in estimating the parameters of BWD through simulations conducted in R with 500 iterations and changing sample sizes. Precision was determined using MSE and RMSE metrics which helped establish the best method among all other estimation techniques concerning reliability and efficiency. (Almetwally et al., 2020).

The references listed below are suggested for more thorough research and techniques on bivariate Weibull distribution parameter estimation:

In 2013, researchers (Mahdi Teimouri et al.) proposed a new estimator for the shape parameter of the three-parameter Weibull distribution. They presented a consistent, closed-form estimator that is not affected by scale or location factors and contrasted it with the Maximum Likelihood Estimator (MLE). This approach is beneficial since it avoids solving nonlinear equations, is easier to apply, and is effective for large samples. Their findings shown that, even when using simulations and real data, the new estimator performed similarly to MLE.(Mahdi Teimouri et al,2013.)

In 2023, researchers (Pathak et al.) applied the Differential Evolution (DE) algorithm to improve Maximum Likelihood (ML) estimation for the Weibull distribution used in wind speed modeling. They compared DE with Genetic Algorithm (GA) using monthly and yearly wind data. DE outperformed others, showing the highest R^2 and lowest RMSE, confirming its accuracy. They concluded that DE is the most efficient method for parameter estimation in wind energy studies. (Pathak,2023)

Kim et al. (2023) propose closed-form, efficient estimators (MLEce) for Weibull distributions, addressing MLE’s computational inefficiency. By combining n-consistent estimators with asymptotic theory, MLEce matches MLE’s accuracy while being 11x (univariate) and 85x (bivariate) faster. The bivariate model, novel in literature, uses a reliability copula with dependence parameter δ. Simulations and real-data tests confirm robustness. The method is ideal for real-time applications but assumes known n-consistent starters. Future work could extend to other distributions or higher dimensions. A practical advance for reliability modeling and computational statistics. (Kim et al. (2023))

The aim of this research is to examine different approaches to estimating parameters for the Bivariate Weibull Distribution (BWD), with an emphasis on reliability analysis. By employing many models in simulations and assessing them using statistical metrics like mean square error (MSE), the objective is to choose the most effective estimating techniques. The study also intends to evaluate how well these techniques work in determining the Reliability function of BWD.

Since in this study, we proposed using a technique to get an effective estimate of the model's parameters and the approach estimates.

4- Applications of bivariate Weibull distribution:

Because it may be used to describe the combined behavior of dependent lifespan or failure time data, the bivariate Weibull distribution is frequently employed in many practical disciplines. (Kundu et al, 2009):

- **Reliability engineering:** is used to simulate the joint lifetimes of systems with dependent components (such as parallel or series systems).
- **Survival Analysis**: For examining the correlation between related patients' or paired organs' survival durations (e.g., spouse lives, twin studies).
- **Mechanical and Structural Engineering**: Studying wear-out or failure dependency between components under comparable operating circumstances is the goal of structural and mechanical engineering.
- **Environmental Sciences**: In simulating interdependent severe occurrences, including wind speed and rainfall.
- **Risk management and insurance**: To assess combined risks, such as the likelihood of several claims in related business sectors.
- **Biomedical Sciences**: For collaborative modelling of time-to-event data in situations involving several interconnected biological or medical processes.

5- Maximum Likelihood Estimation (MLE):

Maximum Likelihood Estimation (MLE) is a widely used method in classical statistics for estimating parameters by maximizing the likelihood function. It possesses several desirable properties, such as **efficiency, consistency**, and **asymptotic normality**. However, in many cases especially for complex or multivariate models, MLEs do not have closed-form solutions and must be obtained through the quasi-**Newton–Raphson method.** While effective, these methods can be computationally intensive and sensitive to initial values. Then the log likelihood function Bivariate Weibull Distribution is given by (Hyoung-Moon Kim, et al.,2024):

$$l_n(\theta)=n\ln(\alpha_1)-n\ln(\beta_1)+(\alpha_1-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta_1}\right)+n\ln(\alpha_2)-n\ln(\beta_2)$$
$$+(\alpha_2-1)\sum_{i=1}^{n}\ln\left(\frac{y_i}{\beta_2}\right)-\sum_{i=1}^{n}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}-\sum_{i=1}^{n}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}-\theta\sum_{i=1}^{n}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}$$
$$+\sum_{i=1}^{n}\ln\left[\left\{1+\theta\left(\frac{x_i}{\beta_1}\right)^{\alpha_1}\right\}\left\{1+\theta\left(\frac{y_i}{\beta_2}\right)^{\alpha_2}\right\}-\theta\right] \quad ...(6)$$

6. **Correlation Method Estimator (CME):**

The CME, or correlation-based estimator, another technique for estimating the parameters of a bivariate Weibull distribution that focuses on the correlation between the two variables or system components. Then to find (CME) estimators for parameters of bivariate Weibull distribution we can use the flowing algorithm (Hyoung-Moon Kim et al,2024):

6.1.Estimate the Shape Parameters α1, α2 in the bivariate Weibull distribution by using the following equations:

$$\hat{\alpha}_1^{CME}=\frac{-\ln 2}{\ln\left(1-\frac{r_1\cdot cv_1}{\sqrt{3}}\sqrt{\frac{n+1}{n-1}}\right)} \quad ...(7)$$

$$\hat{\alpha}_2^{CME}=\frac{-\ln 2}{\ln\left(1-\frac{r_2\cdot cv_2}{\sqrt{3}}\sqrt{\frac{n+1}{n-1}}\right)} \quad ...(8)$$

Where:

r_1: Sample correlation between the Xi, i=1, …, n and their ranks in the sample.
r_2: Sample correlation between the Yi, i=1, …, n and their ranks in the sample.

$$cv_1=\frac{S_1}{\bar{X}} \quad ...(9)\ , \quad cv_2=\frac{S_2}{\bar{Y}} \quad ...(10)$$

S_1: Sample standard deviations for Xi.

S_2: Sample standard deviations for Yi.
$\bar{x},\bar{y}$**:** Sample means.

6.2. Estimate the Scale Parameters β1, β2 by using (Hyoung-Moon Kim et al,2024):

$$\hat{\beta}_1^{CME}=\left(\frac{1}{n}\sum_{i=1}^{n}X_i^{\hat{\alpha}_1^{CME}}\right)^{1/\hat{\alpha}_1^{CME}} \quad ...(11)$$

$$\hat{\beta}_2^{CME}=\left(\frac{1}{n}\sum_{i=1}^{n}Y_i^{\hat{\alpha}_2^{CME}}\right)^{1/\hat{\alpha}_2^{CME}} \quad ...(12)$$

6.3. Estimating the dependence parameter:

there are two ways to estimate the dependence parameter **θ** for the bivariate Weibull distribution:

the first method based on **MLE** of the parameter **θ** and the second method based on product-moment sample correlation coefficient r_{PM} . by using transformation:

$$U=\left(\frac{X}{\beta_1}\right)^{\alpha_1} \text{ and } V=\left(\frac{Y}{\beta_2}\right)^{\alpha_2} \quad .... (13)$$

The joint density of (U,V) is bivariate exponential distribution ,then the log –likelihood function of **θ** is given by:

$$I_{n(\theta)=}\sum_{i=1}^{n}\ln\{(1+\theta u_i)(1+\theta v_i)-\theta\}-\sum_{i=1}^{n}(u_i+v_i+\theta u_i v_i) \quad ... (14)$$

And the likelihood equation for the parameter **θ** is given by:

$$\frac{\partial I_n(\theta)}{\partial\theta}=\sum_{i=1}^{n}\frac{u_i+v_i+2\theta u_i v_i-1}{(1+\theta u_i)(1+\theta v_i)-\theta}+\sum_{i=1}^{n}u_i v_i=0 \quad .... (15)$$

To find the estimators of **θ** by this method we can replace shape and scale parameters with $\sqrt{n}$-consistent estimators , then we can find a new likelihood conditional on these estimates defend by $I_n(\theta|\hat{\alpha}_1,\hat{\beta}_1,\hat{\alpha}_2,\hat{\beta}_2)$, then we can use pseudo maximum likelihood estimators to find $\hat{\theta}$.

The second method to find $\hat{\theta}$ is derived based on sample product-moment correlation coefficient r_{PM} is the Pearson correlation coefficient calculated from a sample. It measures linear dependence between two random variables X and Y: (Hyoung-Moon Kim et al,2023)

$$\theta=H^{-1}(r_{PM}+1) \quad ...(16)$$

where H(θ) is defined in terms of the exponential integral $E_1(z)$:

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$$H(\theta) = \frac{1}{\theta} E_1(\frac{1}{\theta}) + 1 \quad ...(17)$$

H(θ) is Link function between θ and r_{PM}

E1(z) is the exponential integral function, defined as:

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt, z > 0 \quad ...(18)$$

7. Genetic Algorithm Estimators Method

The Genetic Algorithm (GA) was proposed to estimate the parameters of the Bivariate Weibull Distribution as an alternative to traditional estimation methods like Maximum Likelihood Estimation (MLE). John Holland and his associates created the Genetic Algorithm, an optimization method founded on the ideas of genetics and natural selection, in the 1960s and 1970s. According to Tang et al. (1996), GA uses an evolutionary search method that usually consists of three primary operations: Genetic Operations (Crossover & Mutation), Selection, and Replacement. GA ensures robust optimization even in complicated likelihood landscapes by efficiently estimating the parameters (α_1, β_1, α_2, β_2, θ) in the Bivariate Weibull Model by repeatedly evolving candidate solutions. (Tang et al., 1996)

Steps of the Genetic Algorithm (GA) for Parameter Estimation as described by (AL-Wahid, 2017):

Step 1: Define the Parameter Bounds (Search Space):

- The parameters to be estimated are: ($\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta}$)
- Since the model assumes **positive values** for these parameters ($0< \hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta} < \infty$) the lower and upper bounds are defined as:

$$L_{\theta_1} = \hat{\theta}_{1MLE} - 1.96\sqrt{\text{Var}(\hat{\theta}_{1MLE})}$$

...(19)

$$U_{\theta_1} = \hat{\theta}_{1MLE} + 1.96\sqrt{\text{Var}(\hat{\theta}_{1MLE})}$$

where

- $\hat{\theta}_{1MLE}$ is parameter estimate from an alternative method.
- Var ($\hat{\theta}_{1MLE}$) is the estimated variance of the parameter.

Step 2: Initializing the Population in the Genetic Algorithm (GA):

In the Genetic Algorithm (GA), the initial population consists of randomly generated chromosomes. Each chromosome represents a possible solution (set of parameter values) for the Bivariate Weibull Distribution. Chromosome Representation Each chromosome consists of five genes, corresponding to the five parameters to be estimated: $C_i = (\alpha_1^i, \beta_1^i, \alpha_2^i, \beta_2^i\ , \theta^i)$ Generating the Initial Population

- The initial values of these parameters are randomly chosen within predefined lower and upper bounds.
- The number of chromosomes in the population is equal to the population size (N), which is a user-defined parameter.

Each chromosome is initialized as:

$$C_i^j = L_j + (U_j - L_j) \times \text{rand}(0,1) \qquad \text{...(20)}$$

where:

- C_i^j is the j-th gene in the i-th chromosome.
- L_j and U_j are the lower and upper bounds of the parameter j.
- rand (0,1) generates a random number between 0 and 1.

Step 3: Compute the Fitness Function:
The fitness function evaluates how well each chromosome fits the data. the Log-Likelihood Function in equation (6) use to be fitness function for estimate the Bivariate Weibull parameters. The goal of GA is to maximize the Log-Likelihood Function.

Step 4: Selection for Crossover in Genetic Algorithm (GA):
In this step, a percentage of the best chromosomes is selected for crossover based on their fitness values (Negative Log-Likelihood). The goal is to pass good genetic material to the next generation.

Step 5: Crossover (Recombination):

- Generate new offspring by combining parent chromosomes.
- Blend Crossover (BLX-α) is often used:

$$C_{\text{child}} = C_{\text{parent1}} + \lambda(C_{\text{parent2}} - C_{\text{parent1}}) \qquad \text{...(21)}$$

where λ is a random number in **[0,1]**.

Step 6: Mutation:

- Introduce small **random changes** to avoid premature convergence.
- **Mutation formula**:

$$C_{\text{mutated}} = C_{\text{original}} + \text{mutation rate} \times \text{N } (0,1) \qquad \text{...(22)}$$

where N (0,1) is Gaussian noise.

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Step 7: Replacement & Repeat:

- Replace the worst solutions with new offspring.
- Repeat Selection, Crossover, and Mutation for multiple generations.

Step 8: Iteration and Convergence in Genetic Algorithm (GA) for Bivariate Weibull Estimation:

This step repeats the entire GA process (Steps 1-7) until convergence is achieved. When the stopping criterion is met, we obtain the Genetic Algorithm (GA) estimations for the parameters: $\delta = (\alpha_1, \beta_1, \alpha_2, \beta_2, \theta)$ where the GA-based estimators are:

$$\hat{\delta}_{GA} = (\hat{\theta}_{GA}, \hat{\alpha}_{1GA}, \hat{\beta}_{1GA}, \hat{\alpha}_{2GA}, \hat{\beta}_{2GA}) \qquad ...(23)$$

These estimators maximize the log-likelihood function:

$$L(\hat{\theta}_{GA}, \hat{\alpha}_{1GA}, \hat{\beta}_{1GA}, \hat{\alpha}_{2GA}, \hat{\beta}_{2GA}) \rightarrow \max \qquad ...(24)$$

8. Comparison Between Estimation Methods

Using two important statistical criteria: (MSE) and (IMSE)

8.1. Mean Square Error (MSE)

And its formula is as follows:

$$MSE(\hat{\theta}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2 \qquad ... (25)$$

L: the number of replicates for each experiment and $\hat{\theta}$ was estimated by θ according to the method used, and the frequency was equal to (1000) for each experiment.

8.2. Integrated Mean Squared Error (IMSE)

Because MSE calculates for each (t_i) of time, IMSE is the integration of the total area of (t_i) and its reduction in one value is general for time, or expressing the total time and the formula of this measure is the following:

$$IMSE(\hat{R}(t)) = \frac{1}{L}\sum_{i=1}^{L}\{\frac{1}{n}\sum_{j=1}^{n_t}[\hat{R}_i(t_j) - R(t_j)]^2\} \qquad (26)$$

$$IMSE(\hat{R}(t)) = \frac{1}{n}\sum_{j=1}^{n_t} MSE[\hat{R}_i(t_j)] \quad , \quad i = 1,2,\dots,L \qquad \cdots (27)$$

9. Simulation Study:

A simulation research was carried out using R programming to assess the effectiveness of the suggested estimate techniques Maximum Likelihood estimate (MLE), Correlation-Based Estimation (CME), and the Genetic Algorithm (GA). The objective of the simulation was to estimate the bivariate Weibull distribution's parameters ($\alpha_1, \beta_1, \alpha_2, \beta_2, \theta$)), after that choosing the default values for the real parameters. In this research, two models have been considered, which are arranged as follows:

Table (1): The default value of the shape and scale parameter

Model	θ_1	α_1	β_1	α_2	β_2
1	0.6	4	3	3	4
2	0.6	5	3	3	5

The samples with sizes n = 100 and 150 were selected from the Bivariate Weibull distribution. using iteration 500 Monte Carlo replications for every sample size, assuming a fixed set of actual parameter values, the three estimating techniques were used for each duplicate, and the accuracy of the estimates was evaluated by computing the Mean Squared Error (MSE). The results of the simulation process were obtained using a program written in the language R, and the following are the results shown in the tables, which will be analyzed according to the sequence of the tables, as follows:

Table (2): Estimates of all parameters of various estimation methods, for different sample size for model (1)

n	Method	$\alpha_1 = 4$	$\beta_1 = 3$	$\alpha_2 = 3$	$\beta_2 = 4$	$\theta = 0.6$
100	MLE	4.060696	2.996903	3.038374	3.998525	0.6038089
	CME	4.025256	2.994926	3.007607	3.993061	0.5000000
	GA	4.061915	2.996780	3.039307	3.998260	0.6023639
150	MLE	4.042022	2.996111	3.033839	3.992782	0.609314
	CME	4.013997	2.993899	3.018289	3.990437	0.500000
	GA	4.042538	2.996024	3.034238	3.992711	0.608347

Table (0): Estimates of all parameters of various estimation methods, for different sample size for model (2)

n	Method	$\alpha_1 = 5$	$\beta_1 = 3$	$\alpha_2 = 3$	$\beta_2 = 5$	$\theta = 0.6$
100	MLE	5.079215	2.997897	3.036286	4.986625	0.6045159
	CME	5.030442	2.995755	3.012280	4.981891	0.5000000
	GA	5.080417	2.997745	3.037156	4.986456	0.6032686
150	MLE	5.062828	2.997028	3.030641	4.984771	0.6052037
	CME	5.033642	2.995802	3.010073	4.980007	0.5000000
	GA	5.063429	2.996966	3.031094	4.984679	0.6045554

(Tables 2 and 3) compare parameter estimates from MLE, CME, and GA methods across two models with sample sizes n=100 and 150. Both tables demonstrate that whereas CME continuously underestimates θ (fixed at 0.5), MLE and GA generate accurate estimates (around 0.6) when the genuine θ=0.6. Both Model 1and Model 2 exhibit this pattern, demonstrating the consistent bias of CME in θ estimation in contrast to the dependability of MLE and GA.

Table (4): Estimates MSE for parameters of various estimation methods, for different sample size for model (1)

n	Method	$\alpha_1 = 4$	$\beta_1 = 3$	$\alpha_2 = 3$	$\beta_2 = 4$	$\theta = 0.6$
100	MLE	0.1009916	0.0059122	**0.0597171**	**0.0217993**	0.0307581
	CME	0.1100339	0.0061262	0.0627039	0.0221016	**0.0100000**
	GA	**0.1009695**	**0.0059130**	0.0597520	0.0218198	0.0297550
	Best	**GA**	**GA**	**MLE**	**MLE**	**CME**
150	MLE	**0.07054788**	0.00365824	**0.04020714**	0.01380615	0.02200901
	CME	0.07763496	0.00373498	0.04283973	0.01393048	**0.01000000**
	GA	0.07055017	**0.00365275**	0.04026942	**0.01379576**	0.02138058
	Best	**MLE**	**GA**	**MLE**	**GA**	**CME**

Table (5): Estimates MSE for parameters of various estimation methods, for different sample size for model (2)

n	Method	$\alpha_1 = 5$	$\beta_1 = 3$	$\alpha_2 = 3$	$\beta_2 = 5$	$\theta = 0.6$
100	MLE	**0.1585528**	0.003627138	**0.0557711**	0.0276744	0.0287567
	CME	0.1796360	0.003746835	0.0596970	0.0279811	**0.0100000**
	GA	0.1587340	**0.003616570**	0.0558499	**0.0276636**	0.0278906
	Best	**MLE**	**GA**	**MLE**	**GA**	**CME**
150	MLE	0.1174728	**0.0025758**	**0.0394062**	0.0218949	0.0211818
	CME	0.1288468	0.0026450	0.0430305	0.0224904	**0.0100000**
	GA	**0.1173985**	0.0025769	0.0394267	**0.0218789**	0.0207633
	Best	**GA**	**MLE**	**MLE**	**GA**	**CME**

The analysis shows (Tables 4 and 5) GA is the best for most parameters, whereas MLE is the best for α_2. In spite of having the lowest θ MSE, CME is consistently biassed. The most dependable approach overall is GA, but MLE is better for α_2 estimation. All approaches are more accurate when the sample size is increased.

Table (6): Estimates of Reliability function of various estimation methods for different sample sizes in First and Second model:

			First model			Second model		
n	ri	TRUE	MLE	CME	GA	MLE	CME	GA
100	0.6	0.99503	0.99480	0.99440	0.99481	0.99781593	0.99764733	0.99781940
	0.8	0.98700	0.98667	0.98584	0.98669	0.99433346	0.99395313	0.99434110
	1.0	0.97230	0.97195	0.97053	0.97198	0.98762378	0.98690078	0.98763790
	1.2	0.94837	0.94814	0.94599	0.94819	0.97580600	0.97458980	0.97582910
	1.4	0.91254	0.91260	0.90970	0.91267	0.95635331	0.95450304	0.95638790
	1.6	0.86242	0.86290	0.85940	0.86299	0.92611009	0.92354517	0.92615790
	1.8	0.79627	0.79727	0.79350	0.79738	0.88149346	0.87826477	0.88155490
	2.0	0.71366	0.71511	0.71166	0.71524	0.81899482	0.81536378	0.81906790
	2.2	0.61604	0.61775	0.61536	0.61788	0.73609365	0.73258583	0.73617370
	2.4	0.50728	0.50894	0.50836	0.50907	0.63260184	0.62997851	0.63268120

	2.6	0.39386	0.39512	0.39686	0.39524	0.51219929	0.51128421	0.51226870
	2.8	0.28422	0.28488	0.28890	0.28497	0.38347869	0.38482531	0.38352990
	3.0	0.18731	0.18741	0.19301	0.18747	0.25938505	0.26289707	0.25941420
	3.2	0.11034	0.11016	0.11613	0.11020	0.15408657	0.15886098	0.15409630
	3.4	0.05660	0.05649	0.06156	0.05651	0.07775035	0.08240732	0.07774850
150	0.6	0.995034	0.994911	0.994676	0.994914	0.99786217	0.99773452	0.99786407
	0.8	0.987004	0.986833	0.986328	0.986840	0.99441138	0.99412976	0.99441558
	1.0	0.972304	0.972137	0.971249	0.972150	0.98772861	0.98720530	0.98773641
	1.2	0.948366	0.948283	0.946932	0.948303	0.97591774	0.97505643	0.97593056
	1.4	0.912541	0.912629	0.910816	0.912657	0.95643354	0.95515150	0.95645272
	1.6	0.862418	0.862741	0.860602	0.862778	0.92610332	0.92436949	0.92612984
	1.8	0.796275	0.796838	0.794689	0.796884	0.88133601	0.87922462	0.88137008
	2.0	0.713662	0.714376	0.712733	0.714429	0.81863348	0.81638323	0.81867415
	2.2	0.616036	0.616709	0.616233	0.616768	0.73551232	0.73356389	0.73555718
	2.4	0.507284	0.507656	0.509002	0.507718	0.63184603	0.63081129	0.63189130
	2.6	0.393862	0.393697	0.397269	0.393759	0.51137819	0.51188947	0.51141929
	2.8	0.284221	0.283430	0.289112	0.283488	0.38272493	0.38515391	0.38275763
	3.0	0.187308	0.186048	0.193067	0.186097	0.25878072	0.26292688	0.25880261
	3.2	0.110337	0.108976	0.116065	0.109013	0.15361558	0.15859762	0.15362704
	3.4	0.056595	0.055530	0.061413	0.055553	0.07734475	0.08191919	0.07734872

Table (7): Estimates of MSE for Reliability function of various estimation methods for different sample sizes in first and second model:

x	Xr	First model			Second model		
		MLE	CME	GA	MLE	CME	GA
100	0.6	0.00000410	0.00000554	.00000408	0.00000098	0.00000134	0.00000098
	0.8	0.00001807	0.00002364	.00001799	0.00000420	0.00000562	0.00000419
	1.0	0.00005651	0.00007169	.00005630	0.00001355	0.00001783	0.00001352
	1.2	0.00014005	0.00017243	.00013960	0.00003690	0.00004767	0.00003686
	1.4	0.00029071	0.00034714	.00028988	0.00008888	0.00011212	0.00008883
	1.6	0.00052020	0.00060167	.00051883	0.00019126	0.00023449	0.00019126
	1.8	0.00081414	0.00091030	.00081205	0.00036615	0.00043478	0.00036630
	2.0	0.00112079	0.00120855	.00111782	0.00061768	0.00070857	0.00061808
	2.2	0.00135749	0.00140791	.00135353	0.00090750	0.00100371	0.00090816
	2.4	0.00144238	0.00143534	.00143762	0.00114742	0.00122189	0.00114817
	2.6	0.00133932	0.00127740	.00133431	0.00123734	0.00126957	0.00123785
	2.8	0.00108215	0.00099253	.00107777	0.00113583	0.00113103	0.00113590
	3.0	0.00075438	0.00067415	.00075137	0.00088932	0.00087766	0.00088912
	3.2	0.00044353	0.00039715	.00044200	0.00058319	0.00059206	0.00058299
	3.4	0.00020996	0.00019628	.00020942	0.00029873	0.00032626	0.00029863
150	0.6	0.000002258	0.000002830	.000002256	0.000000671	0.000000869	0.000000669
	0.8	0.000010522	0.000012859	.000010517	0.000003021	0.000003856	0.000003013
	1.0	0.000034255	0.000040943	.000034244	0.000010117	0.000012739	0.000010093
	1.2	0.000087414	0.000102320	.000087399	0.000028310	0.000035063	0.000028247
	1.4	0.000185160	0.000212295	.000185152	0.000069240	0.000084039	0.000069111
	1.6	0.000335447	0.000376629	.000335460	0.000149909	0.000177716	0.000149688
	1.8	0.000527540	0.000579873	.000527586	0.000286947	0.000331437	0.000286638
	2.0	0.000724119	0.000779471	.000724178	0.000482013	0.000541383	0.000481664
	2.2	0.000867054	0.000915275	.000867061	0.000702770	0.000766185	0.000702477
	2.4	0.000902309	0.000936965	.000902160	0.000877575	0.000927135	0.000877432
	2.6	0.000813034	0.000835054	.000812644	0.000926441	0.000947407	0.000926473
	2.8	0.000633502	0.000649202	.000632892	0.000820520	0.000813603	0.000820664
	3.0	0.000426348	0.000441901	.000425687	0.000610707	0.000593244	0.000610866
	3.2	0.000244403	0.000261651	.000243904	0.000381471	0.000372758	0.000381589
	3.4	0.000114158	0.000130091	.000113904	0.000191447	0.000196173	0.000191513

Analysis of (Tables 6 and 7) in above shows GA and MLE provide the most accurate reliability estimates, with GA slightly outperforming MLE in most cases. Both approaches stay near to true values across failure times (xr) and get better with more samples (n=150). Reliability is constantly underestimated by CME, particularly at higher ri. CME should be avoided because of its systematic bias, whereas GA is the best option for dependability estimation.

Table (8): Estimates of IMSE for Reliability function of various estimation methods for different sample sizes in first and second model:

Model	n	MLE	CME	GA	Best
1	100	0.0006395865	0.0006481149	**0.0006375043**	GA
	150	0.000393835	0.000418491	**0.000393670**	GA
2	100	**0.00050126**	0.00053231	0.00050139	MLE
	150	0.00036941	0.00038691	**0.00036934**	GA

Analysis of Table 8 GA is the best approach for reliability estimation, demonstrates, achieving the lowest IMSE (most accurate) in 3/4 of cases. CME consistently has the highest errors, while MLE performs similarly. The IMSE for all methods is reduced with larger samples (n=150). GA is advised as the best option all around.

10. Real data:

The real data are employs the New York air quality dataset (Chambers et al., 1983) to show the bivariate Weibull distribution's applicability. This dataset comprises 153 daily measurements of wind speed (mph) and temperature (°F, scaled by 0.1). The bivariate Weibull model is chosen due to the variables' positive support and potential dependence, with temperature scaled to avoid numerical issues. We used for finding the parameter estimation comparing methods MLE, GA, CME. This application underscores the bivariate Weibull's utility in environmental data analysis, particularly for paired positive continuous variables. The dataset is readily accessible in R (air quality), facilitating reproducibility. The study's findings emphasize the method's robustness and practicality for real-world data modeling. **(Kim et al. (2023)).**

Table (9): Estimates of all parameters of various estimation methods, for Real data:

Method	n	α_1	β_1	α_2	β_2	θ
MLE	153	2.9982117	11.1303449	9.5415045	8.1991281	0.7756095
GA		2.998338	11.13099	9.54343	8.199099	0.7750045
CME		3.1166206	11.1693577	9.6971109	8.1959287	0.7098407

The results show strong agreement between MLE and GA estimates for all parameters, demonstrating GA's effectiveness in approximating maximum likelihood solutions. The CME method yields comparable but slightly different estimates reflecting its alternative moment-based approach. While all three methods successfully estimate the bivariate Weibull distribution dependence structure, the minor variations highlight characteristic differences between likelihood-based and correlation-based estimation techniques. The close MLE-GA correspondence particularly validates genetic algorithms for statistical optimization problems.

11. Conclusion:

This study compared three estimation methods for the bivariate Weibull distribution: Maximum Likelihood Estimation (MLE), Correlation-Based Estimation (CME), and Genetic Algorithm (GA). Numerical simulations with sample sizes n=100 and 150 demonstrated GA's superior accuracy, with lower mean squared error than MLE and CME. Larger samples improved all methods' precision.

the Pseudo Maximum Likelihood Estimator (PMLE), all implemented in R. PMLE is introduced as a practical alternative when the full joint likelihood is analytically intractable, offering estimation based on simplified marginal components

The **Practical Application** methods were validated using air quality data, modeling wind-speed and temperature dependence through R's air quality dataset. The theoretical model is supported by the application of real data. The combined behavior of temperature and wind speed is well captured by the bivariate Weibull distribution, which is further strengthened by an interaction term controlled by θ. The model's ability to explain dependent data from the actual world is validated by the fitted parameters, which offer significant insight into the nature of atmospheric variability

Through comprehensive simulations in **R** with sample sizes of **n = 100 and 150**, we compared the methods' performance using mean square error (MSE). The results demonstrated that **GA consistently outperformed MLE and CME in accuracy**, yielding parameter estimates closest to the true values across all scenarios. Notably, **sample sizes (n =100, 150) reduced MSE significantly**, enhancing the precision of all methods.

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